Basic Probability 2

Stochastics

Illés Horváth

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Distributions

A random variable is a random number.

The distribution of a random variable is the possible values and their probabilities.

For a discrete variable X, the distribution can be described by the values

$$\mathbb{P}(X = k) = p_k, \quad k = 0, 1, \dots$$

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For a discrete variable X, the distribution can be described by the values

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For a continuous random variable X, its distribution is described by the cumulative distribution function (cdf)

$$F(x) = \mathbb{P}(X < x),$$

or by the probability density function

$$f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}.$$

Bernoulli distribution

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X has *Bernoulli distribution* with parameter p, or $X \sim \mathrm{I}(p)$ for short, if

$$\mathbb{P}(X = 1) = p, \qquad \mathbb{P}(X = 0) = 1 - p.$$

X can be interpreted as the result of a single trial where the probability of success is p.

$$\mathbb{E}(X) = p.$$

Discrete uniform distribution

X has *discrete uniform distribution* with parameter n, or $X \sim \mathrm{DU}(n)$, if

$$\mathbb{P}(X = k) = \frac{1}{n}, \quad k = 1, \dots, n.$$

X can be interpreted as the result of a roll with a fair n-sided die.

$$\mathbb{E}(X) = \frac{1+n}{2}.$$

Geometric distribution

X has *geometric distribution* with parameter p, or $X \sim \mathrm{GEO}(p)$, if

$$\mathbb{P}(X = k) = p(1 - p)^{k-1}, \quad k = 1, 2, \ldots$$

X can be interpreted as the number of trials needed to get the first success, if each trial is independent and successful with probability p.

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Example. We keep rolling a fair 6-sided die until we roll a 6. The total number of rolls has distribution GEO(1/6).

Pessimistic geometric distribution

Y has *pessimistic geometric distribution* with parameter p, or $Y \sim \mathrm{PGEO}(p)$, if

$$\mathbb{P}(Y = k) = p(1-p)^k, \quad k = 0, 1, \ldots$$

Y can be interpreted as the number of trials *before* the first success, if each trial is independent and successful with probability p (so not counting the actual successful trial).

$$\mathbb{E}(Y) = \frac{1}{p} - 1.$$

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$$\mathbb{E}(Y) = \frac{1}{p} - 1.$$

If $X \sim \mathrm{GEO}(p)$, then $Y = X - 1 \sim \mathrm{PGEO}(p)$ and vice versa.

Binomial distribution

X has *binomial distribution* with parameters n and p, or $X \sim \mathrm{BIN}(n, p)$, if

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

$\left(\binom{n}{k} = \frac{n!}{k!(n-k)!}\right)$
X can be interpreted as the number of successful trials from n trials if each trial is independent and successful with probability p.

$$\mathbb{E}(X) = np.$$

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$$\mathbb{E}(X) = np.$$

Example. If we flip a fair coin 10 times, the number of heads has distribution $\mathrm{BIN}(10, 1/2)$.

Poisson distribution

X has *Poisson distribution* with parameter λ, or $X \sim \mathrm{POI}(\lambda)$, if

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \ldots$$

X can be used to model the number of *rare events*, where the average number of events is λ. We assume that the events are coming from many different independent sources, and the contribution of each source is small.

$$\mathbb{E}(X) = \lambda.$$

Poisson distribution

Example. We know that on a low traffic road, on average 2 cars pass per minute. Then the number of cars passing in a given 1 minute interval has distribution $\mathrm{POI}(2)$.
Consider the following. The number of potential cars that can pass on the road in a given time interval is large, but the probability that a given car will pass there is very small; still, overall, the average number of cars is 2.

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Consider the following. The number of potential cars that can pass on the road in a given time interval is large, but the probability that a given car will pass there is very small; still, overall, the average number of cars is 2.

Example. The number of fires in a city in a given year has Poisson distribution.

Example. The number of packages arriving to an internet server in a given time interval has Poisson distribution.

Example. The number of errors in a book has Poisson distribution.

Uniform distribution

X has *uniform distribution* over the interval $[a, b]$, or $X \sim \mathrm{U}(a, b)$, if its pdf is

$$f(x) = \frac{1}{b-a}, \quad x \in [a, b].$$

This is a continuous distribution that can be used to model a random point within an interval.

$$\mathbb{E}(X) = \frac{a+b}{2}.$$

Exponential distribution

X has *exponential distribution* with parameter λ, or $X \sim \mathrm{EXP}(\lambda)$, if its pdf is

$$f(x) = \lambda e^{-\lambda x}, \quad x \in [0, \infty).$$

This is a continuous distribution that can be used to model the time of the first occurrence of a random event.

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λ is the *rate* or density, so if λ is larger, X is typically smaller.

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Examples. The time we have to wait for the first car to pass / first fire in a city / first request arriving to an internet server etc. has exponential distribution.

Normal distribution

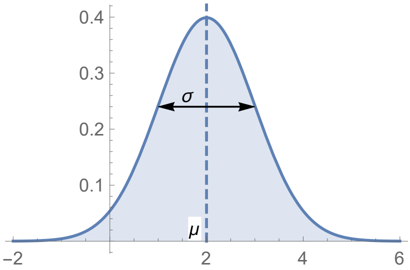
X has *normal distribution* with parameters μ and σ, or $X \sim \mathrm{N}(\mu, \sigma)$, if its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

This is a continuous distribution that can be used to model a random number that is typically close to its mean, but can also be further away with smaller probability.

$$\mathbb{E}(X) = \mu, \qquad \mathbb{D}(X) = \sigma.$$

Normal distribution



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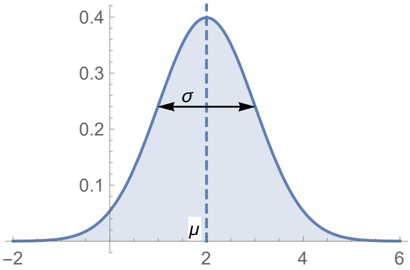

Example. The height (or other physical attributes) of a random person in a population can be modeled by normal distribution.
Example. Measurement error is often modeled by normal distribution.

Pareto distribution

X has *Pareto distribution* with parameters $A > 0$ (scale) and $\alpha > 0$ (shape), or $X \sim \mathrm{Pareto}(A, \alpha)$, if its cdf is

$$F(x) = 1 - \left(\frac{A}{x}\right)^{\alpha}, \quad x \geq A.$$

This is a continuous distribution that can be used to model random numbers where extremely large values may also occur.

$$\mathbb{E}(X) = \begin{cases} \frac{\alpha A}{\alpha - 1} & \alpha > 1 \\ \infty & 0 < \alpha \leq 1 \end{cases}$$

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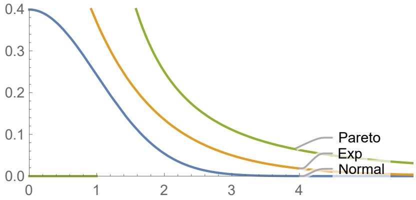
Example. The size of cities can be modeled by Pareto distribution.
Example. The distribution of wealth within a society can be modeled by Pareto distribution.

Decay of distributions

The pdf of the normal distribution decays very rapidly, the exponential distribution is still fast, Pareto is slower.



Two-dimensional distributions

Assume two discrete random variables X and Y are given on the same probability space. Then their *joint 2-dimensional distribution* can be described by

$$p_{k,l} = \mathbb{P}(X = k, Y = l), \quad k = 0, 1, \ldots, l = 0, 1, \ldots,$$

where the $p_{k,l}$'s are nonnegative and add up to 1.

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where the $p_{k,l}$'s are nonnegative and add up to 1.

The *marginal distributions* of X and Y can be computed as

$$\mathbb{P}(X = k) = \sum_{l=0}^{\infty} \mathbb{P}(X = k, Y = l),$$

$$\mathbb{P}(Y = l) = \sum_{k=0}^{\infty} \mathbb{P}(X = k, Y = l).$$

Two-dimensional distributions

If X and Y are both continuous, then their *joint cumulative distribution function* is

$$F(x, y) = \mathbb{P}(X < x, Y < y),$$

and their *joint probability density function* is

$$f(x, y) = \frac{\partial^2 \mathbb{P}(X < x, Y < y)}{\partial x \partial y}.$$

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The *marginal distributions* of X and Y have pdf's

$$f_X(x) = \int_{\mathbb{R}} f(x, y)\mathrm{d}y, \qquad f_Y(y) = \int_{\mathbb{R}} f(x, y)\mathrm{d}x.$$

Conditional distributions

If X and Y are discrete, then the *conditional distribution* of X assuming $Y = l$ is

$$\mathbb{P}(X = k | Y = l) = \frac{\mathbb{P}(X = k, Y = l)}{\mathbb{P}(Y = l)}.$$

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If X and Y are continuous, then the conditional distribution of X assuming $Y = y$ has pdf

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

Independence

X and Y are *independent random variables* if the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for any $A, B \subseteq \mathbb{R}$.

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Theorem

X and Y are independent if and only if

$$\mathbb{P}(X = k, Y = l) = \mathbb{P}(X = k)\mathbb{P}(Y = l) \quad \forall k, l = 0, 1, \dots$$

for X, Y discrete and

$$f(x, y) = f_X(x) f_Y(y) \quad \forall x, y \in \mathbb{R}$$

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for X, Y continuous.

If X and Y are independent, then $\mathbb{D}^2(X + Y) = \mathbb{D}^2(X) + \mathbb{D}^2(Y)$.

Expectation of functions

For a function $g(x, y)$,

$$\mathbb{E}(g(X,Y)) = \begin{cases} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} g(k,l)\mathbb{P}(X=k, Y=l) & \text{for } X, Y \text{ discrete,} \\ \int_{\mathbb{R}} \int_{\mathbb{R}} g(x,y) f(x,y) \mathrm{d}x \mathrm{d}y & \text{for } X, Y \text{ continuous.} \end{cases}$$

Covariance

The *covariance* of X and Y is

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

Covariance measures linear dependence between X and Y.

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If $\mathrm{Cov}(X, Y) < 0$, then if X is large, then Y will be typically small and vice versa.

If X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$ (but the reverse is not true in general).

Correlation

The *correlation* of X and Y is

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\mathbb{D}(X)\mathbb{D}(Y)}.$$

Theorem (Cauchy-Schwarz inequality)

$$-1 \leq \mathrm{Corr}(X, Y) \leq 1.$$

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The correlation is basically a normalized version of the covariance.

$\mathrm{Corr}(X, Y) = 1$ corresponds to full linear dependence between X and Y.

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The distribution of X is $\mathrm{BIN}(5, 1/2)$, and so

$$\mathbb{P}(X = 2) = \binom{5}{2} \left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{2}\right)^{5-2} = \frac{10}{32}.$$

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Solution. Let X denote the number of shark attacks in the given year. Then the question is $\mathbb{P}(X \leq 1)$.

The distribution of X is $X \sim \mathrm{POI}(2.3)$, so

$$\mathbb{P}(X \leq 1) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) = \underbrace{\frac{2.3^0}{0!}}_{=1} e^{-2.3} + \frac{2.3^1}{1!} e^{-2.3} \approx 0.331.$$

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Solution. Let X denote the number of errors on the selected page. Since there are 1000 errors total in the book, and each error has a probability of $1/500$ to appear on that page, the distribution of X is $X \sim \mathrm{BIN}(1000, 1/500)$, and

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) =$$

$$1 - \binom{1000}{0}\left(\frac{1}{500}\right)^0\left(\frac{499}{500}\right)^{1000} - \binom{1000}{1}\left(\frac{1}{500}\right)^1\left(\frac{499}{500}\right)^{999}.$$

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$$\mathbb{P}(Y \geq 2) = 1 - \mathbb{P}(Y = 0) - \mathbb{P}(Y = 1) =$$
$$1 - \frac{2^0}{0!}e^{-2} - \frac{2^1}{1!}e^{-2}.$$

So which one is correct?

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So which one is correct?

Computing them numerically:

$$\mathbb{P}(X \geq 2) \approx 0.594265,$$
$$\mathbb{P}(Y \geq 2) \approx 0.593994.$$

Problem 6

In fact, this can be stated as a theorem.

Theorem

Let $n \to \infty$ and $p_n \to 0$ such that $np_n \to \lambda > 0$, and let $X_n \sim BIN(n, p_n)$ and $Y \sim POI(\lambda)$. Then

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathbb{P}(Y = k), \qquad \forall k \geq 0$$

(We also say that X_n converges in distribution to Y, or $X_n \xrightarrow{d} Y$.)

Problem 8

Assume that the age of a light bulb X (measured in 100 hours) has an exponential distribution such that $\mathbb{P}(X > 10) = 0.8$. Calculate the parameter of the exponential distribution and the mean of X.

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Solution. Let λ denote the parameter of the exponential distribution. Then its cdf is

$$F(x) = 1 - e^{-\lambda x},$$

and

$$\mathbb{P}(X > 10) = 1 - \mathbb{P}(X < 10) = 1 - F(10) = e^{-10\lambda} = 0.8,$$

from which $\lambda = -\log(0.8)/10 \approx 0.0223$, and

$$\mathbb{E}(X) = \frac{1}{\lambda} \approx 44.8 \quad \text{(in 100 hours)}.$$

Problem 10

In a class of 120 students, Stochastics and Calculus marks are as follows:

$C\backslash S$	1	2	3	4	5
1	1	2	2	1	4
2	2	4	4	8	2
3	4	8	8	12	8
4	5	4	6	9	6
5	0	6	4	6	4

We pick a student at random; let X denote his Stochastics mark and Y his Calculus mark.

a) $\mathbb{P}(\text{the student failed at least one of the courses}) =?$
b) $\mathbb{E}(X) =?$
c) $\mathbb{E}(X|Y \geq 4) =?$
d) Are X and Y independent?
e) $\mathrm{Cov}(X, Y) =?$

Problem 10

Solution.

a) A total of 22 students failed at least one of the courses (marked with red in the table), so

$$\mathbb{P}(\text{the student failed at least one of the courses}) = \frac{21}{120}.$$

$C \backslash S$	1	2	3	4	5
1	1	2	2	1	4
2	2	4	4	8	2
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4	5	4	6	9	6
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Problem 10

(b) We need to compute the marginal distribution of X.

$C \backslash S$	1	2	3	4	5
1	1	2	2	1	4
2	2	4	4	8	2
3	4	8	8	12	8
4	5	4	6	9	6
5	0	6	4	6	4
	12	24	24	36	24

So the marginal distribution of X is

k	1	2	3	4	5
$\mathbb{P}(X = k)$	$\frac{12}{120}$	$\frac{24}{120}$	$\frac{24}{120}$	$\frac{36}{120}$	$\frac{24}{120}$

and $\mathbb{E}(X) = \frac{12}{120} \cdot 1 + \frac{24}{120} \cdot 2 + \frac{36}{120} \cdot 3 + \frac{24}{120} \cdot 4 + \frac{24}{120} \cdot 5 = 3.2.$

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(c) We need to compute the conditional distribution of X assuming $Y \geq 4$. First note that $\mathbb{P}(Y \geq 4) = \frac{50}{120}$.

$C \backslash S$	1	2	3	4	5
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5	0	6	4	6	4
	5	10	10	15	10

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$C \backslash S$	1	2	3	4	5
4	5	4	6	9	6
5	0	6	4	6	4
	5	10	10	15	10

So the conditional distribution of X assuming $Y \geq 4$ is

k	1	2	3	4	5
$\mathbb{P}(X = k \mid Y \geq 4)$	$\frac{5}{50}$	$\frac{10}{50}$	$\frac{10}{50}$	$\frac{15}{50}$	$\frac{10}{50}$

and $\mathbb{E}(X \mid Y \geq 4) = \frac{5}{50} \cdot 1 + \frac{10}{50} \cdot 2 + \frac{10}{50} \cdot 3 + \frac{15}{50} \cdot 4 + \frac{10}{50} \cdot 5 = 3.2.$

Problem 10

(d) No, for example

$$\mathbb{P}(X = 1, Y = 5) = 0 \neq \mathbb{P}(X = 1)\mathbb{P}(Y = 5) = \frac{12}{120} \cdot \frac{20}{120}.$$

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$$\mathbb{P}(X=1, Y=5) = 0 \neq \mathbb{P}(X=1)\mathbb{P}(Y=5) = \frac{12}{120} \cdot \frac{20}{120}.$$

(e)

$$\mathbb{E}(X) = 3.2,$$
$$\mathbb{E}(Y) = 3.25,$$
$$\mathbb{E}(XY) = \frac{1}{120} \cdot 1 \cdot 1 + \frac{2}{120} \cdot 1 \cdot 2 + \cdots + \frac{4}{120} \cdot 5 \cdot 5 = 10.4,$$

so

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0,$$

even though X and Y are not independent.

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$$\mathbb{E}(X) = 3.2,$$
$$\mathbb{E}(Y) = 3.25,$$
$$\mathbb{E}(XY) = \frac{1}{120} \cdot 1 \cdot 1 + \frac{2}{120} \cdot 1 \cdot 2 + \dots + \frac{4}{120} \cdot 5 \cdot 5 = 10.4,$$

so

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0,$$

even though X and Y are not independent.

Bonus question: how was the table designed?